



8.4 Special linear maps and linear operators

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Introduction

- In this topic, we will
 - Define a number of special linear maps and operators
 - The zero map
 - The identity operator
 - Diagonal maps
 - Super-diagonal maps
 - Sub-diagonal maps
 - Tri-diagonal maps
 - Also introduce the delay and advance operators for discrete-time signals





The zero map and operator

- The zero map $O: \mathcal{U} \rightarrow \mathcal{V}$ maps every vector in $\mathcal{U}$ onto the zero vector $\mathbf{0}_{\mathcal{V}}$ in $\mathcal{V}$

Theorem

The zero map is linear.

Proof:

$$O(\alpha \mathbf{u}_1 + \beta \mathbf{u}_2) = \mathbf{0}_{\mathcal{V}}$$

$$\alpha O\mathbf{u}_1 + \beta O\mathbf{u}_2 = \alpha \mathbf{0}_{\mathcal{V}} + \beta \mathbf{0}_{\mathcal{V}} = \mathbf{0}_{\mathcal{V}}$$

Since these are equal, this map is linear. ■

- The zero map $O: \mathbf{F}^n \rightarrow \mathbf{F}^m$ is represented by an $m \times n$ matrix of all zeros, and is often denoted $O_{m,n}$





The identity operator

- The identity operator $I: \mathcal{U} \rightarrow \mathcal{U}$ maps every vector in $\mathcal{U}$ onto itself.

Theorem

The identity operator is linear.

Proof:

$$I(\alpha \mathbf{u}_1 + \beta \mathbf{u}_2) = \alpha \mathbf{u}_1 + \beta \mathbf{u}_2$$

$$\alpha I\mathbf{u}_1 + \beta I\mathbf{u}_2 = \alpha \mathbf{u}_1 + \beta \mathbf{u}_2$$

Since these are equal, this operator is linear. ■





The identity operator

- The identity operator $I: \mathbf{F}^n \rightarrow \mathbf{F}^n$ is represented by an $n \times n$ matrix of all zeros except for all ones on the diagonal, and is denoted I_n
- For example,

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- Note that the columns of the identity operator are the n canonical basis vectors of $\mathbf{F}^n$





Diagonal maps and operators

- Any $m \times n$ matrix with zeros everywhere except perhaps on the diagonal called a diagonal map or diagonal operator
 - Generally, we only use diagonal operators

- If $D = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix}$, then $D\mathbf{u} = \begin{pmatrix} \lambda_1 u_1 \\ \lambda_2 u_2 \\ \lambda_3 u_3 \\ \lambda_4 u_4 \end{pmatrix}$





Super-diagonal maps and operators

- Any $m \times n$ matrix with zeros everywhere except perhaps on the super-diagonal is called a super-diagonal map or super-diagonal operator

- If $S = \begin{pmatrix} 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, then $S\mathbf{u} = \begin{pmatrix} \lambda_1 u_2 \\ \lambda_2 u_3 \\ \lambda_3 u_4 \\ 0 \end{pmatrix}$

- Note that if all the super-diagonal entries are one, this has an effect of shifting the entries of $\mathbf{u}$ up by one





Sub-diagonal maps and operators

- Any $m \times n$ matrix with zeros everywhere except perhaps on the sub-diagonal is called sub-diagonal map or sub-diagonal operator

- If $S = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \end{pmatrix}$, then $S\mathbf{u} = \begin{pmatrix} 0 \\ \lambda_1 u_1 \\ \lambda_2 u_2 \\ \lambda_3 u_3 \end{pmatrix}$

- Note that if all the sub-diagonal entries are one, this has an effect of shifting the entries of $\mathbf{u}$ down by one





Tri-diagonal maps and operators

- Any $m \times n$ matrix with zeros everywhere except perhaps on the super-diagonal, the diagonal and the sub-diagonal is called a tri-diagonal map or tri-diagonal operator

- If $S = \begin{pmatrix} \lambda_{1,1} & \lambda_{1,2} & 0 & 0 & 0 \\ \lambda_{2,1} & \lambda_{2,2} & \lambda_{2,3} & 0 & 0 \\ 0 & \lambda_{3,2} & \lambda_{3,3} & \lambda_{3,4} & 0 \\ 0 & 0 & \lambda_{4,3} & \lambda_{4,4} & \lambda_{4,5} \\ 0 & 0 & 0 & \lambda_{5,4} & \lambda_{5,5} \end{pmatrix}$, then $S\mathbf{u} = \begin{pmatrix} \lambda_{1,1}u_1 + \lambda_{1,2}u_2 \\ \lambda_{2,1}u_1 + \lambda_{2,2}u_2 + \lambda_{2,3}u_3 \\ \lambda_{3,2}u_2 + \lambda_{3,3}u_3 + \lambda_{3,4}u_4 \\ \lambda_{4,3}u_4 + \lambda_{4,4}u_4 + \lambda_{4,5}u_5 \\ \lambda_{5,4}u_4 + \lambda_{5,5}u_5 \end{pmatrix}$





Operators on discrete-time signals

- For the vector space of discrete-time signals, two common operators are the delay and advance operators
- If $x = (x[0], x[1], x[2], x[3], \dots)$, then the action of the delay operator D is defined as

$$Dx = (0, x[0], x[1], x[2], \dots)$$

and the action of the advance operator E is defined as

$$Ex = (x[1], x[2], x[3], x[4], \dots)$$





Summary

- Following this topic, you now
 - Are aware of some of the more common linear maps and operators
 - Know the delay and advance operators for discrete-time signals





References

- [1] https://en.wikipedia.org/wiki/Identity_function
- [2] https://en.wikipedia.org/wiki/Zero_matrix
- [3] https://en.wikipedia.org/wiki/Diagonal_matrix
- [4] https://en.wikipedia.org/wiki/Tridiagonal_matrix





Acknowledgments

None so far.





Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using Consolas. Mathematical equations are prepared in MathType by Design Science, Inc. Examples may be formulated and checked using Maple by Maplesoft, Inc.

The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

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